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# Biorthogonal coupling coefficients of $u_q(n)$

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**Abstract.** The coupling (Wigner–Clebsch–Gordan) coefficients and their isofactors for the unitary quantum algebras $u_q(n)$ with the repeating irreducible representations in the coproduct decomposition are considered. Generalizing the $U(n)$ case, the biorthogonal systems of the $u_q(n)$ isofactors with the dual multiplicity labels are constructed by means of the recoupling technique in terms of the isofactors with simpler multiplicity structure. A first construction, correlated with the inverted Littlewood–Richardson rules, gives the bilinear combinations of isofactors after applying the proportionality of the $q$-recoupling (Racah) coefficients to the boundary $q$-isofactors. An alternative recursive construction gives the nonorthogonal $q$-isofactors satisfying the most elementary boundary conditions and proportional to the $u_q(n-1)$ recoupling coefficients for some less restricted values of parameters. Some multiplicity-free and more general $u_q(n)$ recoupling coefficients are found, the blocks (bilinear combinations) of which (equal to the resubducing coefficients of the complementary chains of $q$-algebras) are proposed to use for the orthonormalization of some $u_q(n)$ biorthogonal isofactors, including the general $u_q(3)$ case.

## 1. Introduction

In recent years, many investigations have been devoted to the representation theory of quantized universal enveloping algebras ($q$-algebras) or quantum groups, which found applications in the theory of quantum integrable systems, statistical mechanics, conformal field theory, and in the phenomenologic models of atomic, molecular and nuclear spectroscopy. This $q$-representation theory is related to non-commutative geometry and the theory of knots and links, but it has its origin in the representation theory of usual Lie algebras. Many analogies between Lie and $q$-algebras are known, which can be extended to some subalgebra chains, branching rules, coupling and recoupling coefficients and the main structures of irreducible tensor calculus.

Different authors considered the coupling (Clebsch–Gordan–Wigner) and recoupling (Racah) coefficients of the multiplicity-free quantum algebra $u_q(2)$. Some aspects of the irreducible tensor calculus for the unitary quantum groups ($q$-algebras) $u_q(n)$ with $n \geqslant 3$ were developed by Biedenharn (1990), Tolstoy (1990), Smirnov *et al* (1991b), Gould *et al* (1992), Gould (1992), Gould and Biedenharn (1992), Klimyk (1992, 1993), Quesne (1992, 1993), Lienert and Butler (1992), Pan and Chen (1993a), Smirnov and Kharitonov (1993a, b). In a previous paper (Ališauskas and Smirnov 1994), the most general multiplicity-free isoscalar factors (isofactors) of $u_q(n)$ coupling coefficients (for coupling an arbitrary and symmetric representations) were derived and explicit expansion of arbitrary (most general) isofactors in terms of the restricted boundary values was proposed. A summary of these

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results, presented in section 2 in slightly modified form, is replenished by some new expressions of the multiplicity-free recoupling coefficients of $u_q(n)$.

The main purpose of this paper is the consideration of the most general coupling coefficients of the quantum algebra $u_q(n)$ with repeating irreducible representations (irreps) in the coproduct decomposition. As in the case of the unitary group $U(n)$, the explicit analytical expressions give, as a rule, only non-orthogonal systems of the $u_q(n)$ isofactors with the multiplicity labels of irreps. In sections 3 and 4, we generalize the approach grounded on the $U(n)$ recoupling technique (cf Ališauskas 1972, 1978, 1983, 1988, Chen 1987) for the explicit construction of the biorthogonal systems of the (dual) isofactors with repeating irreps, respectively, formed by the bilinear combinations of isofactors and the isofactors satisfying the dual boundary conditions. For this purpose in the first situation, we use the relation between isofactors and recoupling coefficients generalized to quantum algebras (cf Ališauskas *et al* 1971, Sullivan 1973, Kramer *et al* 1981), wheras in the second situation we use the boundary properties of the auxiliary isofactors, also demonstrating their proportionality to recoupling coefficients. For special values of parameters (and $n \geqslant 4$), we inevitably have a special case and this atypical construction of the $u_q(4)$ isofactors with distinctive boundary behaviour is presented in the appendix, together with different applications of the technique introduced.

For coupling an arbitrary and a two parametric (covariant or mixed tensor) irrep of $u_q(n)$, we express some overlaps of the coupled non-orthonormal states in terms of the presented $q$-recoupling (Racah) coefficients, equivalent to the resubducing coefficients of some complementary (Quesne 1992, Smirnov and Tolstoy 1992, Malashin *et al* 1992, 1994) chains of the quantum subalgebras. This system of overlaps is sufficient for the explicit orthogonalization of all the $u_q(3)$ isofactors to the paracanonical scheme (cf Ališauskas 1988, 1990).

## 2. Defining relations, summary of previous results and multiplicity-free recoupling coefficients

We fix the same commutation and comultiplication rules for the generators of the unitary quantum algebra $u_q(n) = U_q(u(n))$ as in the previous paper (Ališauskas and Smirnov 1994). The quantum algebra $u_q(n)$ is a deformation of the $u(n)$ enveloping algebra. It is defined by generators $e_{i,i+1}, e_{i+1,i}$, $i = 1, 2, \ldots, n-1$, and $h_i = e_{ii}$, $i = 1, 2, \ldots, n$, which satisfy the commutation relations

$$[h_i, h_j] = 0 \qquad (i \neq j) \tag{2.1a}$$

$$e_{ij+1} = [e_{ij}, e_{ij+1}]_q \equiv e_{ij}e_{jj+1} - qe_{jj+1}e_{ij} \qquad (i < j) \tag{2.1b}$$

$$e_{j+1i} = [e_{j+1j}, e_{ji}]_{q^{-1}} \equiv e_{j+1j}e_{ji} - q^{-1}e_{ji}e_{j+1j} \qquad (i < j) \tag{2.1c}$$

$$[h_i, e_{jk}] = \delta_{ij}e_{ik} - \delta_{ik}e_{ji} \tag{2.1d}$$

$$[e_{ik}, e_{ki}] = [h_i - h_k] \tag{2.1e}$$

and the Serre identities

$$e_{ik}e_{kl}^2 - [2]e_{kl}e_{ik}e_{kl} + e_{kl}^2e_{ik} = 0 \tag{2.2a}$$

($i < k < l$ or $i > k > l$), which may be written in terms of the $q$-deformed commutators

$$[[e_{ik}, e_{kl}]_q, e_{kl}]_{q^{-1}} = [[e_{ik}, e_{kl}]_{q^{-1}}, e_{kl}]_q = 0. \tag{2.2b}$$

Here and below $[x]$ is a $q$-number

$$[x] = -[-x] = (q^x - q^{-x})/(q - q^{-1}) \tag{2.3a}$$

and, for $x$ integer, $[x]!$ is a $q$-factorial

$$[x]! = [x][x-1]\cdots[2][1] \tag{2.3b}$$

with $[0]! = 1$ and $[-n]! = \infty$. We use the coproduct rules

$$\Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i \tag{2.4a}$$

$$\Delta(e_{ii+1}) = e_{ii+1} \otimes q^{1/2(h_i-h_{i+1})} + q^{-1/2(h_i-h_{i+1})} \otimes e_{ii+1} \tag{2.4b}$$

$$\Delta(e_{i+1,i}) = e_{i+1i} \otimes q^{1/2(h_i-h_{i+1})} + q^{-1/2(h_i-h_{i+1})} \otimes e_{i+1i}. \tag{2.4c}$$

The irreducible representations (irreps) of $u_q(n)$ are denoted by the highest-weight $\lambda_{(n)}$ or the partition (Young scheme) $[\lambda_{1n}, \lambda_{2n}, \ldots, \lambda_{nn}]$. The canonical (Gelfand–Tsetlin) basis states may be denoted (Tolstoy 1990) as

$$|\lambda\rangle_q \equiv \left| \begin{array}{ccccccccc} \lambda_{1n} & & \lambda_{2n} & & \cdots & & \lambda_{n-1n} & & \lambda_{nn} \\ & \lambda_{1n-1} & & \lambda_{2n-1} & & \cdots & & \lambda_{n-1n-1} & \\ & & \cdots & & \cdots & & \cdots & & \\ & & & \lambda_{12} & & \lambda_{22} & & & \\ & & & & \lambda_{11} & & & & \end{array} \right\rangle_q \tag{2.5}$$

where $\lambda_{ij}$ $(1 \leqslant i \leqslant j \leqslant n)$ satisfy the standard inequalities

$$\lambda_{ij+1} \geqslant \lambda_{ij} \geqslant \lambda_{i+1,j+1}.$$

### 2.1. *Matrix elements of generator powers, isofactors with one of irreps symmetric and boundary expansion of general isofactors*

As in Ališauskas and Smirnov (1994), we take $q$ real and, of course, not a root of unity. In the previous paper the following expression for the matrix elements of the elementary $u_q(n)$ generator powers in the Gelfand–Tsetlin basis was derived:

$$\left\langle \begin{array}{c} \lambda \\ \mu' \\ \nu \end{array} \middle| e^p_{k-1,k} \middle| \begin{array}{c} \lambda \\ \mu \\ \nu \end{array} \right\rangle_q = \left\langle \begin{array}{c} \lambda \\ \mu \\ \nu \end{array} \middle| e^p_{k,k-1} \middle| \begin{array}{c} \lambda \\ \mu' \\ \nu \end{array} \right\rangle_q$$
$$= \frac{[p]!\ (d_{k-1}[\mu]\, d_{k-1}[\mu'])^{1/2}\ S_{k,k-1}[\lambda;\mu]\ S_{k-1,k-2}[\mu';\nu]}{S^2_{k-1,k-1}[\mu';\mu]\ S_{k,k-1}[\lambda;\mu']\ S_{k-1,k-2}[\mu;\nu]} \tag{2.6}$$

where

$$S_{n,m}[\lambda;\mu] = \left( \frac{\prod_{1\leqslant i\leqslant j\leqslant m}[\lambda_i - \mu_j - i + j]!}{\prod_{1\leqslant j<i\leqslant n}[\mu_j - \lambda_i + i - j - 1]!} \right)^{1/2}$$
$$d_n[\lambda] = \prod_{1\leqslant i<j\leqslant n} [\lambda_i - \lambda_j - i + j] \tag{2.7}$$

and $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n]$, $\mu$, $\nu$ are partitions, satisfying the usual betweenness conditions and $\mu_i$, $\nu_j$ denoting the separate rows.

In the previous paper, the following two classes of expressions for the multiplicity-free isofactors appearing as the reduced matrix elements of the symmetric tensor operators of $u_q(k)$ were also derived. In the first case, we obtained

$$\left[ \begin{array}{ccc} \lambda & p & \lambda' \\ \mu & \overline{p} & \mu' \end{array} \right]^{(k)}_q = q^{\varrho^{(k)}} N_k \left( \begin{array}{cc} \lambda & \lambda' \\ \mu & \mu' \end{array} \right) \left( [p-\overline{p}]!\ d_k[\lambda']\, d_{k-1}[\mu] \right)^{1/2}$$
$$\times \sum_\sigma (-1)^{\sum_{i=1}^{k-1}(\sigma_i-\mu_i)} \frac{q^{-(p-\overline{p}+1)\sum_{j=1}^{k-1}(\sigma_j-\mu_j)}\ d_{k-1}[\sigma]\ S^2_{k,k-1}[\lambda';\sigma]}{S^2_{k-1,k-1}[\sigma;\mu]\ S^2_{k-1,k-1}[\mu';\sigma]\ S^2_{k,k-1}[\lambda;\sigma]} \tag{2.8}$$

where $k-1$ summation parameters $\sigma_i$ $(1 \leqslant i \leqslant k-1)$ accept values

$$\max(\mu_i, \lambda'_{i+1}) \leqslant \sigma_i \leqslant \min(\mu'_i, \lambda_i)$$

and

$$N_k \begin{pmatrix} \lambda & \lambda' \\ \mu & \mu' \end{pmatrix} = \frac{S_{k-1,k-1}[\mu';\mu]\; S_{k,k-1}[\lambda;\mu]}{S_{k,k}[\lambda';\lambda]\; S_{k,k-1}[\lambda';\mu']} \tag{2.9}$$

$$Q^{(k)} = \tfrac{1}{2}\sum_{i<j}\Delta_i\Delta_j - \tfrac{1}{2}\sum_{i<j}\overline{\Delta}_i\overline{\Delta}_j + \tfrac{1}{2}\sum_{i=1}^{k-1}\overline{\Delta}_i(\mu_i - i + 1) - \tfrac{1}{2}\sum_{j=1}^{k}\Delta_j(\lambda_j - j + 1)$$
$$+\tfrac{1}{2}(p-\overline{p})\left(\sum_{i=1}^{k}\lambda_i - \sum_{j=1}^{k-1}\mu_j\right) \tag{2.10}$$

$$\Delta_i = \lambda'_i - \lambda_i \qquad \overline{\Delta}_j = \mu'_j - \mu_j \qquad p = \sum_i \Delta_i \qquad \overline{p} = \sum_j \overline{\Delta}_j.$$

In the second case, $k$ different expressions for the same isofactors (with $i = 1, 2, \ldots, k$) were obtained, namely

$$\begin{bmatrix} \lambda & p & \lambda' \\ \mu & \overline{p} & \mu' \end{bmatrix}_q^{(k)} = N_k^{-1}\begin{pmatrix} \lambda & \lambda' \\ \mu & \mu' \end{pmatrix}\left(\frac{d_k[\lambda']\; d_{k-1}[\mu]}{[p-\overline{p}]!}\right)^{1/2}$$
$$\times \sum_{\sigma_j, j\neq i} \frac{(-1)^{\phi_i}\, q^{R_i^{(k)}}\, d_k[\sigma]\, S^2_{k,k-1}[\sigma;\mu]}{S^2_{k,k-1}[\sigma;\mu']\, S^2_{k,k}[\sigma;\lambda]\, S^2_{k,k}[\lambda';\sigma]} \tag{2.11}$$

where $k-1$ summation parameters $\sigma_j$ $(1 \leqslant j \leqslant k, j \neq i)$ accept values

$$\max(\mu'_j, \lambda_j) \leqslant \sigma_j \leqslant \min(\mu_{j-1}, \lambda'_j),$$

the factors with $\sigma_i$ are omitted in $d_k[\sigma]$ and $S^2_{k,k'}[\cdots;\cdots]$ on the r.h.s. and

$$\phi_i = \sum_{i=1}^{i-1}(\mu'_j - \mu_j + \lambda_j) + \sum_{i=i+1}^{k}\lambda'_j - \sum_{l=1, j\neq i}^{k}\sigma_j \tag{2.12}$$

$$R_i^{(k)} = \tfrac{1}{2}\sum_{j<j'}\Delta_j\Delta_{j'} - \tfrac{1}{2}\sum_{j<j'}\overline{\Delta}_j\overline{\Delta}'_j + \tfrac{1}{2}\sum_{l=1}^{k}\Delta_j(\lambda'_j - j) - \tfrac{1}{2}\sum_{i=1}^{k-1}\overline{\Delta}_j(\mu'_j - j)$$
$$-\tfrac{1}{2}(p-\overline{p})\left(\sum_{j}^{k}\lambda'_j + \sum_{j}^{k-1}\mu'_j - 2i + 1\right) + (p-\overline{p})\sum_{i=1, j\neq i}^{k}\sigma_j. \tag{2.13}$$

The formulas (2.8) and (2.11) display quite different behaviour for extreme values of some parameters. Their special versions, also presented in Ališauskas and Smirnov (1994), will be used in sections 3 and 4.

The general $u_q(n) \supset u_q(n-1)$ isofactors appear as the expansion coefficients (in the sum over the multiplicity labels of the repeating irreps of subalgebra) of the $u_q(n)$ coupling coefficients in terms of the $u_q(n-1)$ coupling coefficients. Using the $q$-binomial expansion of coproducts (Smirnov *et al* 1991b) and acting with operator $e^p_{n,n-1}$ into the coupled semi-maximal state of irrep $\lambda$ of $u_q(n)$ with fixed multiplicity label, the following recursive expansion of arbitrary isofactors of $u_q(n) \supset u_q(n-1)$ in terms of restricted (boundary) isofactors was obtained by *Ališauskas and Smirnov (1994)*:

$$\left[\begin{array}{ccc} \lambda' & \lambda'' & {}^{\rho}\lambda \\ \mu' & \mu'' & {}^{\overline{\rho}}\mu \end{array}\right]_q^{(n)} \left\langle \begin{array}{c} \lambda \\ \mu \\ \nu \end{array} \middle| e_{nn-1}^{p'+p''} \middle| \begin{array}{c} \lambda \\ \overline{\lambda} \\ \nu \end{array} \right\rangle_q =$$

$$= \sum_{\nu',\nu'',\tilde{\mu}',\tilde{\mu}'',\overline{\overline{\rho}},\tilde{\overline{\rho}}} \frac{[p'+p'']!}{[p']![p'']!} q^{\mathcal{Q}'_{(n)}} \left[\begin{array}{ccc} \mu' & \mu'' & {}^{\overline{\rho}}\mu \\ \nu' & \nu'' & {}^{\overline{\overline{\rho}}}\nu \end{array}\right]_q^{(n-1)} \left\langle \begin{array}{c} \lambda' \\ \mu' \\ \nu' \end{array} \middle| e_{nn-1}^{p'} \middle| \begin{array}{c} \lambda' \\ \tilde{\mu}' \\ \nu' \end{array} \right\rangle_q$$

$$\times \left\langle \begin{array}{c} \lambda'' \\ \mu'' \\ \nu'' \end{array} \middle| e_{nn-1}^{p''} \middle| \begin{array}{c} \lambda'' \\ \tilde{\mu}'' \\ \nu'' \end{array} \right\rangle_q \left[\begin{array}{ccc} \lambda' & \lambda'' & {}^{\rho}\lambda \\ \tilde{\mu}' & \tilde{\mu}'' & {}^{\tilde{\overline{\rho}}}\overline{\lambda} \end{array}\right]_q^{(n)} \left[\begin{array}{ccc} \tilde{\mu}' & \tilde{\mu}'' & {}^{\tilde{\overline{\rho}}}\overline{\lambda} \\ \nu' & \nu'' & {}^{\overline{\overline{\rho}}}\nu \end{array}\right]_q^{(n-1)} \tag{2.14}$$

where

$$\overline{\lambda} = [\lambda_1, \lambda_2, \ldots, \lambda_{n-1}] \qquad p' = \sum_{i=1}^{n-1} (\tilde{\mu}'_i - \mu'_i) \qquad p'' = \sum_{i=1}^{n-1} (\tilde{\mu}''_i - \mu''_i)$$
$$\mathcal{Q}'_{(n)} = \tfrac{1}{2} p' \left( 2 \sum_{j=1}^{n-1} \mu''_j - \sum_{j=1}^{n} \lambda''_j - \sum_{j=1}^{n-2} \nu''_j \right) - \tfrac{1}{2} p'' \left( 2 \sum_{j=1}^{n-1} \mu'_j - \sum_{j=1}^{n} \lambda'_j - \sum_{j=1}^{n-2} \nu'_j \right). \tag{2.15}$$

The first or the last isofactor on the r.h.s. of (2.14) may be simplified choosing $\nu = [\mu_1, \ldots, \mu_{n-2}]$ or $\nu = [\lambda_2, \ldots, \lambda_{n-1}]$. Then it is possible to see that the sum over partitions $\nu', \nu''$ and the multiplicity label $\overline{\overline{\rho}}$ on the r.h.s. of (2.14), together with $q$-phase factor is proportional to the stretched $9j$-coefficient, multiplied by an elementary isofactor of $u_q(n-1)$

$$\left(\begin{array}{ccc} \mu' & \mu'' & {}^{\overline{\rho}}\mu \\ p' & p'' & p \\ \tilde{\mu}' & \tilde{\mu}'' & {}^{\tilde{\overline{\rho}}}\overline{\lambda} \end{array}\right)_q^{(n-1)} \left[\begin{array}{ccc} \mu'' & p & \overline{\lambda} \\ \nu & 0 & \nu \end{array}\right]_q^{(n-1)}. \tag{2.16}$$

In the semistretched case (for $\lambda_n = \lambda'_n + \lambda''_n = 0$), i.e. for those terms in the $u_q(n)$ coproduct which also appeared in the $u_q(n-1)$ coproduct decomposition in the case of irreps denoted by the same partitions, the auxiliary isofactor (the last one in (2.14)) is unity and the general semistretched isofactor of $u_q(n)$ should be proportional† to the $u_q(n-1)$ recoupling coefficient (2.16), with $\tilde{\mu}' = \overline{\lambda}' \equiv \lambda'$ and $\tilde{\mu}'' = \overline{\lambda}'' \equiv \lambda''$.

### 2.2. *Some multiplicity-free recoupling coefficients*

The simple expressions of $u_q(n-1)$ recoupling coefficients as $q$ $6j$-coefficients were proposed for $n = 2$ by Kirillov and Reshetikhin (1988), Nomura (1989), Kachurik and Klimyk (1990), Smirnov *et al* (1991a). For the most general recoupling coefficients between the schemes (1 2)3 and 1(2 3), we use the following notation:

$$U_n\left(j_1\, j_2\, {}^{\rho_{12,3}}j^{\rho_{1,23}}\, j_3;\, {}^{\rho_{12}}j_{12}\, {}^{\rho_{23}}j_{23}\right)_q = \left\{\begin{array}{cc|c} j_1 & j_2 & {}^{\rho_{12}}j_{12} \\ j_3 & {}^{\rho_{12,3}}j^{\rho_{1,23}} & {}^{\rho_{23}}j_{23} \end{array}\right\}_q^{(n)}. \tag{2.17}$$

Taking into account above-mentioned property of the semistretched $u_q(n)$ isofactors, in analogy with appendix B of Ališauskas *et al* (1972), we may suppose that some recoupling

† Unfortunately, the factor $[2i+1]$ is omitted in the denominator under the square root of (4.8) of Ališauskas and Smirnov (1994), used in the corresponding formulas (4.5) and (4.10) for the semistretched and stretched isofactors of $u_q(3)$.

coefficients between the schemes (1 2)3 and 1(2 3) are proportional to isofactors (2.11). In the case of the partitions $\lambda, \lambda', \mu, \mu'$, each restricted to $n$ rows, we may write

$$U_{n+1}(p' \mu \lambda' p;\, \lambda \mu')_q \begin{bmatrix} p' & \mu' & \lambda' \\ 0 & \mu' & \mu' \end{bmatrix}_q^{(n+1)} = \begin{bmatrix} \lambda & p & \lambda' \\ \mu & p & \mu' \end{bmatrix}_q^{(n+1)} \begin{bmatrix} p' & \mu & \lambda \\ 0 & \mu & \mu \end{bmatrix}_q^{(n+1)} \begin{bmatrix} \mu & p & \mu' \\ \mu & p & \mu' \end{bmatrix}_q^{(n+1)}. \tag{2.18}$$

In this case the $u_q(n+1)$ recoupling coefficient coincides with the $u_q(n)$ recoupling coefficient, the last isofactor on the r.h.s. is equal to 1 and the first isofactor on the r.h.s. may be expressed by means of (2.11) with $i = 1$. These results, together with the symmetry relation (3.24) of Ališauskas and Smirnov (1994) applied to remaining two isofactors, allow us to present the following expression for the multiplicity-free $u_q(n)$ recoupling coefficients:

$$U_n(p' \mu \lambda' p;\, \lambda \mu')_q = (-1)^{\psi} U_n(p \lambda'^* \mu^* p';\, \lambda^* \mu'^*)_q \tag{2.19a}$$

$$= (-1)^{\mu_2 - \mu_2' - \lambda_2 + \sum_{j=3}^{n} \lambda_j'} \frac{S_{n,n}[\lambda';\lambda]\, S_{n,n}[\lambda';\mu']}{S_{n,n}[\mu';\mu]\, S_{n,n}[\lambda;\mu]} \left(d_n[\lambda]\, d_n[\mu']\right)^{1/2}$$
$$\times \sum_{\sigma} (-1)^{\sum_j \sigma_j} \frac{d_{n-1}[\sigma]\, S^2_{n,n-1}[\lambda;\sigma]\, S^2_{n,n-1}[\mu';\sigma]}{S^2_{n,n-1}[\mu;\sigma]\, S^2_{n,n-1(1)}[\lambda';\sigma]} \tag{2.19b}$$

where

$$S^{-2}_{n,n-1(1)}[\lambda';\sigma] = \frac{\prod_{1 \leqslant j < i-1 \leqslant n-1} [\sigma_j - \lambda_i' + i - j - 2]!}{\prod_{1 \leqslant i \leqslant j+1 \leqslant n} [\lambda_i' - \sigma_j - i + j + 1]!}$$

(cf (2.21) of Ališauskas and Smirnov 1994) and $n-1$ summation parameters $\sigma_j$ ($1 \leqslant j \leqslant n-1$) accept values

$$\max(\mu_{j+1}, \lambda_{j+1}) \leqslant \sigma_j \leqslant \min(\mu_j, \lambda'_{j+1}).$$

The symmetry relation (2.19$a$) (with a phase $\psi = 0$ for $u_q(2)$ and $u_q(3)$), together with interchange $p \leftrightarrow p'$, $\lambda \leftrightarrow \lambda^* = [-\lambda_n, \ldots, -\lambda_2, -\lambda_1]$, $\mu \leftrightarrow \lambda'^*$, $\lambda' \leftrightarrow \mu^*$, $\mu' \leftrightarrow \mu'^*$, substitution $\sigma \to \sigma^* = [-\sigma_{n-1}, \ldots, -\sigma_2, -\sigma_1]$ and relations

$$S_{n,n}[\lambda^*;\mu^*] = S_{n,n}[\mu;\lambda] \qquad S_{n,n-1}[\lambda^*;\mu^*] = S^{-1}_{n,n-1}[\lambda;\mu]$$

allows us to obtain an expression with different summation intervals. Of course, the recoupling coefficients presented above form the complete recoupling matrices.

We may also express the following recoupling coefficients without summation:

$$U_n(\mu p' \lambda' p;\, \lambda [p'+p])_q = \left( \begin{bmatrix} \mu & p+p' & \lambda' \\ \mu & 0 & \mu \end{bmatrix}_q^{(n+1)} \right)^{-1} \times \begin{bmatrix} \mu & p' & \lambda \\ \mu & 0 & \mu \end{bmatrix}_q^{(n+1)} \begin{bmatrix} \lambda & p & \lambda' \\ \mu & 0 & \mu \end{bmatrix}_q^{(n+1)} \tag{2.20a}$$

$$= \left( \frac{[p']!\,[p]!\, d_n[\lambda]}{[p'+p]!} \right)^{1/2} \frac{S_{n,n}[\lambda';\mu]}{S_{n,n}[\lambda';\lambda]\, S_{n,n}[\lambda;\mu]} \tag{2.20b}$$

using equations (3.10) and (3.22) of Ališauskas and Smirnov (1994).

### 3. Recoupling technique and the bilinear combinations of isofactors

*Some general aspects of the biorthogonal system concept in the non-multiplicity-free group* representation theory were considered by Ališauskas (1987, 1988). The biorthogonal systems of the $u_q(n)$ isofactors (for the coupling coefficients with the repeating irreducible representations in the coproduct decomposition) may be expressed by means of the recoupling technique, similarly as in the $U(n)$ case (Ališauskas 1978, 1983, 1988). The different recoupled structures are associated either with the bilinear combinations of isofactors, or in contrast, with the isofactors, satisfying the dual boundary conditions.

*The first complete system of the $u_q(n) \supset u_q(n-1)$ isofactors may be selected from the* following recursive relation between the isofactors and recoupling coefficients:

$$\sum_{\rho} \begin{bmatrix} \lambda' & \lambda'' & {}^{\rho}\lambda \\ \mu' & \mu'' & {}^{\overline{\rho}}\mu \end{bmatrix}_q^{(n)} U_n\big(\lambda' \lambda''^{(-)}\, {}^{\rho}\lambda\, \lambda''_1;\, {}^{\bar{\rho}}\Lambda\, \lambda''\big)_q$$

$$= \sum_{\kappa,\overline{\rho},\nu,\bar{\bar{\rho}}} \begin{bmatrix} \lambda''^{(-)} & \lambda''_1 & \lambda'' \\ \kappa & \overline{p} & \mu'' \end{bmatrix}_q^{(n)} \begin{bmatrix} \lambda' & \lambda''^{(-)} & {}^{\bar{\rho}}\Lambda \\ \mu' & \kappa & {}^{\bar{\bar{\rho}}}\nu \end{bmatrix}_q^{(n)}$$

$$\times \begin{bmatrix} \Lambda & \lambda''_1 & \lambda \\ \nu & \overline{p} & \mu \end{bmatrix}_q^{(n)} U_{n-1}\big(\mu'\, \kappa\, {}^{\overline{\rho}}\mu\, \overline{p};\, {}^{\bar{\bar{\rho}}}\nu\, \mu''\big)_q \tag{3.1}$$

where, without loss of generality, the parameters $\lambda'_n$ and $\lambda''_n$ in partitions $\lambda'$ and $\lambda''$, respectively, are chosen to be equal to 0. The partition $\lambda''^{(-)} = [\lambda''_2, \lambda''_3, \ldots, \lambda''_n]$ is obtained from $\lambda''$ after deleting its first parameter $\lambda''_1$ (which labels the symmetric irreps in the first and last isofactors on the r.h.s.). The external multiplicity labels $\rho, \overline{\rho}, \bar{\bar{\rho}}$ may denote the non-orthogonal coupled states of the repeating irreps, but in the sums they should be substituted by the couples of the dual labels (written as subscript and superscript, respectively). The first isofactor on the r.h.s. may be expressed according (2.8) without sum and the second by means of (2.14) (taking into account its semistretched case) or for the two parametric irrep $\lambda'' = [\lambda''_1, \lambda''_2, 0]$ (and in the general $u_q(3)$ case) by means of a specified equation, (2.11), (cf subsection 3.4 of Ališauskas and Smirnov 1994). The last isofactor on the r.h.s. of (3.1) may be expressed by means of (2.8) or (2.11).

In the case of the semistretched coupling of the auxiliary coproduct $\lambda' \otimes \lambda''^{(-)}$ to $\Lambda$ (i.e., for $\Lambda_n = 0$), the recoupling (Racah) coefficient of $u_q(n)$ on the l.h.s. of (3.1) may be expressed (without use of the $R$-matrices) in terms of trivial sum of isofactors

$$U_n\big(\lambda' \lambda''^{(-)}\, {}^{\rho}\lambda\, \lambda''_1;\, {}^{\bar{\rho}}\Lambda\, \lambda''\big)_q \equiv \big\langle \lambda' \lambda''^{(-)} ({}^{\bar{\rho}}\Lambda)\, \lambda''_1;\, \lambda \mid \lambda';\, \lambda''^{(-)} \lambda''_1(\lambda'')\, {}^{\rho}\lambda \big\rangle_q$$

$$= \left( \begin{bmatrix} \Lambda & \lambda''_1 & \lambda \\ \Lambda & 0 & \lambda \end{bmatrix}_q^{(n)} \right)^{-1} \sum \begin{bmatrix} \lambda''^{(-)} & \lambda''_1 & \lambda'' \\ \lambda''^{(-)} & 0 & \lambda''^{(-)} \end{bmatrix}_q^{(n)}$$

$$\times \begin{bmatrix} \lambda' & \lambda'' & {}^{\rho}\lambda \\ \lambda' & \lambda''^{(-)} & {}^{\bar{\rho}}\Lambda \end{bmatrix}_q^{(n)} \begin{bmatrix} \lambda' & \lambda''^{(-)} & {}^{\bar{\rho}}\Lambda \\ \lambda' & \lambda''^{(-)} & {}^{\bar{\rho}}\Lambda \end{bmatrix}_q^{(n)} \tag{3.2a}$$

*and is proportional to the $u_q(n)$ boundary isofactor*

$$U_n\big(\lambda' \lambda''^{(-)}\, {}^{\rho}\lambda\, \lambda''_1;\, {}^{\bar{\rho}}\Lambda\, \lambda''\big)_q$$

$$= q^{\widetilde{R}_{(n)}} \left( \frac{\prod_{i=2}^{n-1} [\lambda''_1 - \lambda''_i - 1 + i]\; M[\lambda]}{[\lambda''_1 - \lambda''_n - 2 + n]!\; M[\Lambda]} \right)^{1/2} \begin{bmatrix} \lambda' & \lambda'' & {}^{\rho}\lambda \\ \lambda' & \lambda''^{(-)} & {}^{\bar{\rho}}\Lambda \end{bmatrix}_q^{(n)} \tag{3.2b}$$

where

$$M[\lambda]=\prod_{i=1}^{k}[\lambda_i-i+k]!/d_k[\lambda] \qquad \text{for } \lambda=[\lambda_1,\lambda_2,\ldots,\lambda_k]$$
$$\widetilde{R}_{(n)}=\tfrac{1}{2}\sum_{j=2}^{n-1}(j-1)(\lambda''_j-\lambda''_{j+1})-\tfrac{1}{2}\sum_{1\leqslant i<j}^{n}\widetilde{\Delta}_i\widetilde{\Delta}_j+\tfrac{1}{2}\sum_{j=1}^{n}\widetilde{\Delta}_j(\Lambda_j-j+1) \tag{3.3}$$

and $\widetilde{\Delta}_j=\lambda_j-\Lambda_j$. Thus, we see that (3.1) and (3.2*b*) together allow us to present the bilinear combination of isofactors as the non-orthonormal isofactor

$$\begin{bmatrix}\lambda' & \lambda'' & {}^{+,-,\rho}\lambda\\ \mu' & \mu'' & {}^{\overline{\rho}}\mu\end{bmatrix}_q^{(n)}=\sum_{\rho'}\begin{bmatrix}\lambda' & \lambda'' & {}^{\rho'}\lambda\\ \mu'_{\max} & \mu''_{\min} & {}_{+,-,\tilde{\rho}}\Lambda\end{bmatrix}_q^{(n)}\begin{bmatrix}\lambda' & \lambda'' & {}_{\rho'}\lambda\\ \mu' & \mu'' & {}^{\overline{\rho}}\mu y\end{bmatrix}_q^{(n)} \tag{3.4}$$

with the unique correspondence between its external multiplicity label $\rho$ (written as a subscript, with $+$ and $-$ indicating the fixed highest- and lowest-weight states in the auxiliary isofactor which appears on the r.h.s. of (3.4) as an expansion coefficient) and a couple $\{\Lambda,\tilde{\rho}\}$, embracing irrep $\Lambda$ of $u_q(n-1)$, together with its multiplicity label $_{+,-,\tilde{\rho}}$, both restricted in accordance with the inverted Littlewood–Richardson rules. In its turn, $\tilde{\rho}$ should be determined by a simpler couple $\{\widetilde{\Lambda},\tilde{\tilde{\rho}}\}$ etc. As it will be demonstrated below, we dispense with this step-by-step procedure, when for some subalgebra $u_q(k)$ we obtain a multiplicity-free coupling and may finish with the orthonormal isofactors of this subalgebra, instead of the type (3.4) expansions. Of course, in the general $u_q(3)$ case and for the two parametric irrep $\lambda''=[\lambda''_1,\lambda''_2,0]$, the external multiplicity label $\rho$ is completely determined by intermediate irrep $\Lambda$.

Suppose that the sum on the r.h.s. of (3.4) include only single term with $\rho'=\{+,-,\rho\}$. Then the dual isofactors (with superscript $^{+,-,\rho}$) should satisfy the boundary condition

$$\begin{bmatrix}\lambda' & \lambda'' & {}^{+,-,\rho}\lambda\\ \mu'_{\max} & \mu''_{\min} & {}_{+,-,\overline{\rho}}\mu\end{bmatrix}_q^{(n)}=\delta_{\{\mu\overline{\rho}\},\rho} \tag{3.5}$$

if $\mu'=\mu'_{\max}=\overline{\lambda}'$, $\mu''=\mu''_{\min}=\lambda''^{(-)}$ and $\{\mu\,\overline{\rho}\}$ accept values in the same region as $\{\Lambda\,\tilde{\rho}\}$, for which the linear combinations (3.1) or (3.4) are linearly independent. For the remaining values of $\mu',\mu'',\mu$ and $\overline{\rho}$, the dual isofactors coincide with the expansion coefficients of arbitrary bilinear combinations of the $u_q(n)$ isofactors in terms of the complete system (3.4). Hence, our multiplicity resolution in isofactors is initiated by fixing two extreme basis states and changing the third state.

The second isofactor on the r.h.s. of (3.1)

$$\begin{bmatrix}\lambda' & \lambda''^{(-)} & {}^{\tilde{\rho}}\Lambda\\ \mu' & \mu'' & {}^{\tilde{\bar{\rho}}}\nu\end{bmatrix}_q^{(n)}$$

may be expressed by means of the correspondingly specified equation (3.1) (with $\lambda''$ substituted by $\lambda''^{(-)}$, $\lambda''^{(-)}$ substituted by $\lambda''^{(-,-)}=[\lambda''_3,\ldots,\lambda''_{n-1}]$, $\lambda''_1$ replaced by $\lambda''_2$, $\Lambda$ replaced by $\widetilde{\Lambda}$ with $\widetilde{\Lambda}_{n-1}=\lambda'_{n-1}$, etc). This way we may finally establish a correspondence between the multiplicity label $\rho$ and the set $\Lambda,\widetilde{\Lambda},\ldots$, which may be presented as a Gelfand–Tsetlin table for irrep $\lambda$ (cf Biedenharn *et al* 1967)

$$\begin{pmatrix}\lambda_1,\lambda_2,\ldots,\lambda_{n-2},\lambda_{n-1},\lambda_n\\ \Lambda_1,\Lambda_2,\ldots,\Lambda_{n-2},\Lambda_{n-1}\\ \widetilde{\Lambda}_1,\widetilde{\Lambda}_2,\ldots,\widetilde{\Lambda}_{n-2}\\ \cdots\cdots\cdots\cdots\\ \lambda'_1\,\lambda'_2\cdots\lambda'_{n-k}\end{pmatrix} \tag{3.6}$$

and includes the variables $\Lambda_i$, $\widetilde{\Lambda}_j$, etc in $k-1$ rows if the partition $\lambda''$ is formed by $k \leqslant n-1$ non-vanishing parts. Otherwise, the inverted Littlewood–Richardson rules (corresponding to the decomposition of $\lambda^* \otimes \lambda''$ to $\lambda'^*$) induce the following restrictions for $\Lambda_i$, $\widetilde{\Lambda}_j$, $\widetilde{\widetilde{\Lambda}}_{j'}$, etc:

$$\begin{gathered}
\lambda_n - \lambda'_n \geqslant \Lambda_{n-1} - \lambda'_{n-1} \\
\lambda_n + \lambda_{n-1} - \Lambda_{n-1} - \lambda'_n \geqslant \Lambda_{n-1} + \Lambda_{n-2} - \widetilde{\Lambda}_{n-2} - \lambda'_{n-1} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\sum_{i=j}^{n} \lambda_i - \sum_{i=j}^{n-1} \Lambda_i - \lambda'_n \geqslant \sum_{i=j-1}^{n-1} \Lambda_i - \sum_{i=j-1}^{n-2} \widetilde{\Lambda}_i - \lambda'_{n-1} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\sum_{i=j}^{n-1} \Lambda_i - \sum_{i=j}^{n-2} \widetilde{\Lambda}_i - \lambda'_{n-1} \geqslant \sum_{i=j-1}^{n-2} \widetilde{\Lambda}_i - \sum_{i=j-2}^{n-3} \widetilde{\widetilde{\Lambda}}_i - \lambda'_{n-2}. \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{gathered} \tag{3.7a}$$

The solution of these inequalities seems rather complicated, with exception of the $k = 2$ case:

$$\sum_{i=j}^{n} \lambda_i - \sum_{i=j}^{n-1} \Lambda_i \geqslant \sum_{i=j-1}^{n-1} \Lambda_i - \sum_{i=j-1}^{n-1} \lambda'_i. \tag{3.7b}$$

The inequalities (3.7*a*) allows us to choose the sufficient diversity of the coupled states but we need to prove that the recursive constructions (3.1) are linearly independent for the different values of our multiplicity labels. The constructions presented in the next section and in the appendix will be useful for this purpose.

Of course, the recursive construction (3.1) of the isofactors is also very complicated. Only for $u_q(3)$ the total number of sums is six. In the irrep $\lambda'' = \varepsilon = [\varepsilon_1, \varepsilon_2]$ case, we have a special possibility of expressing the coupled non-orthonormal states, i.e. the bilinear combinations of isofactors

$$\sum_\rho \begin{bmatrix} \lambda' & \varepsilon & {}^\rho\lambda \\ \overline{\lambda}' & \varepsilon_2 & \Lambda \end{bmatrix}_q^{(n)} \begin{bmatrix} \lambda' & \varepsilon & {}^\rho\lambda \\ \overline{\lambda}' & \varepsilon_2 & \mu \end{bmatrix}_q^{(n)} \tag{3.8a}$$

in terms of the bilinear combinations of the recoupling coefficients

$$\sum_\rho U_n\big(\lambda' \varepsilon_2 {}^\rho\lambda\, \varepsilon_1;\ \Lambda\, \varepsilon\big)_q\ U_n\big(\lambda' \varepsilon_2 {}^\rho\lambda\, \varepsilon_1;\ \mu\, \varepsilon\big)_q \tag{3.8b}$$

which will be considered in section 5.

## 4. Recoupling technique for the boundary expansion of $u_q(n)$ isofactors

Let us consider an alternative construction to (3.1) of the non-orthonormal $u_q(n)$ isofactors with the multiplicity label $\rho$ and the coupling diversity generated by the different values of the intermediate irrep $\Lambda$ and the multiplicity label $\rho'$ from an auxiliary (corresponding to $u_q(n-1)$) coproduct $\Lambda \otimes \gamma$ decomposition. Denoting some irreps as mixed tensor representations (e.g. $[\gamma,0,-p]$, $[\lambda,-h]$, $[\Lambda,-h]$) or contravariant symmetric irreps $[\dot{0},-p]$, we write

$$\begin{bmatrix} \lambda' & [\gamma,0,-p] & {}^{\rho,+,+}[\lambda,-h] \\ \mu' & [\kappa,-r] & {}_{\overline{\rho},+,+}\mu \end{bmatrix}_q^{(n)} = \left( \begin{bmatrix} \lambda' & [\dot{0},-p] & [\Lambda,-h] \\ \Lambda & 0 & \Lambda \end{bmatrix}_q^{(n)} \right)^{-1}$$

$$\times \sum_{\overline{p},\widetilde{\gamma},\widetilde{\Lambda},\overline{\rho}'} \left[ \begin{matrix} [\dot{0},-p] & \gamma & [\gamma,0,-p] \\ [\dot{0},-\overline{p}] & \widetilde{\gamma} & [\kappa,-r] \end{matrix} \right]_q^{(n)} \left[ \begin{matrix} \lambda' & [\dot{0},-p] & [\Lambda,-h] \\ \mu' & [\dot{0},-\overline{p}] & \widetilde{\Lambda} \end{matrix} \right]_q^{(n)}$$

$$\times \left[ \begin{matrix} [\Lambda,-h] & \gamma & {}^{\rho'}[\lambda,-h] \\ \widetilde{\Lambda} & \widetilde{\gamma} & {}^{\overline{\rho}'}\mu \end{matrix} \right]_q^{(n)} U_{n-1}\big(\mu', [\dot{0},-\overline{p}],_{\overline{\rho},+,+}\mu^{\overline{\rho}'}, \widetilde{\gamma}; \widetilde{\Lambda}, [\kappa,-r]\big)_q. \tag{4.1}$$

The partitions $\lambda'$, $\mu'$, $\Lambda$ include no more as $n-1$ parts each and the partitions $\kappa$, $\gamma$, $\widetilde{\gamma}$, respectively, include no more as $n-2$ parts. Here the last parameters of irreps $\widetilde{\Lambda}$ and $\mu$ may be negative, respectively, $\widetilde{\Lambda} \geqslant -h$ and $\mu \geqslant -h$. After the correlated addition of $p$ to the all parameters in the square brackets denoting irreps and in $\Lambda$, $\mu$, $\widetilde{\Lambda}$, this notation turns into the usual one, in terms of partitions (e.g. $[\gamma,0,-p]$ turns into $[\gamma_1+p,\ldots,\gamma_{n-2}+p,p,0]$, $[\lambda,-h]$ turns into $[\lambda_1+p,\ldots,\lambda_{n-1}+p,p-h]$ and $\Lambda$ turns into $[\Lambda_1+p,\ldots,\Lambda_{n-1}+p]$ with $p \geqslant h$). The multiplicity label $\overline{\rho}$ corresponds to the decomposition of the coproduct $\mu' \otimes [\kappa,-r]$ to $\mu$ in the $u_q(n-1)$ case and $\overline{\rho}'$ corresponds to the decomposition of $\widetilde{\Lambda} \otimes \widetilde{\gamma}$ to $\mu$. The last isofactor on the r.h.s. of (4.1) is semistretched (cf the last but one isofactor on the r.h.s. of (3.1)); the remaining three isofactors are multiplicity-free and may be expressed using the symmetries and expressions presented at the end of section 3 of Ališauskas and Smirnov 1994 (two of them without sum). Again, the general equation (4.1) is sufficiently simple only in the $n = 3$ case when the last isofactor on the r.h.s. is multiplicity-free and semistretched (cf subsection 3.4 of Ališauskas and Smirnov 1994) and the total number of sums is six.

We see that the construction (4.1) is possible only for $h \geqslant 0$; for the boundary values of parameters it turns into

$$\left[ \begin{matrix} \lambda' & [\gamma,0,-p] & {}^{\rho,+,+}[\lambda,-h] \\ \mu' & [\gamma,0] & {}_{\overline{\rho},+,+}\lambda \end{matrix} \right]_q^{(n)} = \delta_{\mu'\Lambda}\delta_{\overline{\rho}\rho'} \tag{4.2}$$

i.e. isofactors (4.1) are linearly independent for the all values of the multiplicity label $\rho$ represented by the couple $\Lambda$ and $\rho'$, where $\rho'$ is the multiplicity label of $\lambda$ in the coproduct $\Lambda \otimes \gamma$ decomposition. For the sake of simplicity, we omitted on the l.h.s. of (4.1) the recoupling coefficient which appears in the overlaps of the non-orthogonal isofactors, namely

$$\sum_{\mu',\kappa,r,\overline{\rho}} \left[ \begin{matrix} \lambda' & [\gamma,0,-p] & {}^{\rho,+,+}[\lambda,-h] \\ \mu' & [\kappa,-r] & {}^{\overline{\rho}}\mu \end{matrix} \right]_q^{(n)} \left[ \begin{matrix} \lambda' & [\gamma,0,-p] & {}^{\tilde{\rho},+,+}[\lambda,-h] \\ \mu' & [\kappa,-r] & {}_{\overline{\rho}}\mu \end{matrix} \right]_q^{(n)}$$

$$= \left( \left[ \begin{matrix} \lambda' & [\dot{0},-p] & [\Lambda,-h] \\ \Lambda & 0 & \Lambda \end{matrix} \right]_q^{(n)} \left[ \begin{matrix} \lambda' & [\dot{0},-p] & [\widetilde{\Lambda},-h] \\ \widetilde{\Lambda} & 0 & \widetilde{\Lambda} \end{matrix} \right]_q^{(n)} \right)^{-1}$$

$$\times \sum_{\bar{\bar{\rho}}} U_n\big(\lambda', [\dot{0},-p], {}^{\bar{\bar{\rho}},+,+}[\lambda,-h]^{\rho'}, \gamma;\ [\Lambda,-h], [\gamma,0,-p]\big)_q$$

$$\times U_n\big(\lambda, [\dot{0},-p], {}_{\bar{\bar{\rho}},+,+}[\lambda,-h]^{\tilde{\rho}'}, \gamma;\ [\widetilde{\Lambda},-h], [\gamma,0,-p]\big)_q \tag{4.3}$$

where, in turn, the external multiplicity label $\tilde{\rho}$ of the l.h.s. is represented by the couple $\widetilde{\Lambda}$ and $\tilde{\rho}'$.

The following particular case of (4.1) is especially important:

$$\left[ \begin{matrix} \lambda' & [\gamma,0,-p] & {}^{\rho,+,+}[\lambda,-h] \\ \mu' & [\kappa,-r] & {}_{\overline{\rho},+,+}\lambda \end{matrix} \right]_q^{(n)} = \frac{(-1)^{\mu'_{n-2}-\Lambda_{n-2}+p}\, q^{Q_{(n)}}\, S_{n,n-1}[\lambda';\Lambda]}{S_{n,n-1}[\lambda';\mu']\, S_{n-1,n-1}[\mu';\Lambda]}$$

$$\times\left(\frac{[p-r]!\,d_{n-1}[\mu']}{[p]!}\prod_{j=1}^{n-1}\frac{[\kappa_j+p+n-1-j]!}{[\gamma_j+p+n-1-j]!}\right)^{1/2}$$

$$\times U_{n-1}\big(\mu',[\dot{0},-p],\,{}_{\overline{p},+,+}\lambda^{\rho'},\gamma;\Lambda,[\kappa,-r]\big)_q \qquad \rho=\{\Lambda,\rho'\} \tag{4.4}$$

where

$$\mathcal{Q}_{(n)}=\tfrac{1}{2}\sum_{1\leqslant i<j}^{n-1}(\mu'_i-\Lambda_i)(\mu'_j-\Lambda_j)+\tfrac{1}{2}\sum_{i=1}^{n-1}(\mu'_i-\Lambda_i)(\mu'_i+n-i)+\tfrac{1}{2}p(n-2)$$
$$-\tfrac{1}{2}\sum_{1\leqslant i<j}^{n-2}(\gamma_i-\kappa_i)(\gamma_j-\kappa_j)+\tfrac{1}{2}\sum_{i=1}^{n-2}(\gamma_i-\kappa_i)(\kappa_i+n-i-1). \tag{4.5}$$

Equation (4.4) is valid not only for $h\geqslant 0$ but also for $\lambda_{n-1}+p\geqslant\lambda'_1$ when it may be derived from an alternative to (4.1) construction (with some symmetries applied.) It turns into (4.2), when $\kappa=\gamma$, $r=0$. Otherwise, in the remaining region (for $h<0$ and $\lambda_{n-1}+p<\lambda'_1$) it gives the expansion coefficients of the once restricted isofactors in terms of the doubly restricted isofactors with the free parameters $\mu'$ accepting the all possible values (which number may exceed the multiplicity of irrep $[\lambda,-h]$ in the coproduct $\lambda'\otimes[\gamma,0,-p]$ decomposition). In its turn, (2.14) together with (4.4) gives the complete boundary expansion of isofactors with the fixed external multiplicity label.

We present here the $u_q(3)\supset u_q(2)$ case of the restricted isofactor (4.4) in terms of the $u_q(2)$ $6j$-coefficients

$$\begin{bmatrix} (a'b') & (a''b'') & {}^{\tilde{I}',-,-}(a\ b) \\ (z')i' & (z'')i'' & (z_0)i_0 \end{bmatrix}^{(3)}_q$$
$$=(-1)^{1/2(a''+a)-\tilde{I}'+2i'}q^{\mathcal{Q}'_{(3)}}\left(\frac{[2\tilde{I}'+1][2i'+1][2i''+1]}{[a''+2z'']!}\right)^{1/2}$$
$$\times\frac{G[a''b''i''z'']\,\Gamma[a'b'\tilde{I}'\tilde{z}']}{\nabla[\frac{1}{2}a''+z'',i',\tilde{I}']\,\Gamma[a'b'i'z']}\begin{Bmatrix} i' & i'' & \frac{1}{2}a \\ \frac{1}{2}a'' & \tilde{I}' & \frac{1}{2}a''+z'' \end{Bmatrix}_q \tag{4.6}$$

in parameters used by127Ališauskas (1988) in the $SU(3)$ case when irreps are labelled as mixed tensor irreps $(a'b')$, $(a''b'')$, $(a\ b)$ with

$$\begin{array}{lllll} a=\lambda_1-\lambda_2 & b=\lambda_2 & \lambda_3=0 & i=\frac{1}{2}(\mu_1-\mu_2) & \\ z=\lambda_2-\frac{1}{2}(\mu_1+\mu_2) & i_0=-z_0=\frac{1}{2}a & i''_0=\frac{1}{2}a'' & z_0=z'+z'+v & (4.7) \\ \tilde{z}'=v+\frac{1}{2}(a''-a)' & v=\frac{1}{3}(a'-b'+a''-b''-a+b) & & & \end{array}$$

(cf the $SU(3)\supset U(2)$ case, see Ališauskas 1978, 1982, 1987, Pluhař 1986). Here the following notation is used:

$$\nabla[a\ b\ c]=\left(\frac{[a+b-c]!\,[a-b+c]!\,[a+b+c+1]!}{[b+c-a]!}\right)^{1/2} \tag{4.8}$$

$$G[a\,b\,i\,z]=\left(\frac{[a+2z]!\,[b-z-i]!\,[b-z+i+1]!}{[b]!\,[a+b+1]!}\right)^{1/2} \tag{4.9}$$

$$\Gamma[a\,b\,i\,z]=\left(\frac{[i+z]!\,[a+z-i]!\,[a+z+i+1]!}{[i-z]!\,[b-z-i]!\,[b-z+i+1]!}\right)^{1/2}. \tag{4.10}$$

In equation (4.6), a $q$ $6j$-coefficient and the following $q$-phase appeared:

$$\mathcal{Q}'_{(3)}=\tfrac{1}{2}i'(i'+1)-\tfrac{1}{2}\tilde{I}'(\tilde{I}'+1)-\tfrac{1}{2}i''(i''+1)+\tfrac{1}{2}i''_0(i''_0+1)+\tfrac{1}{2}(a''+2z'')(\tfrac{1}{3}a+b+3). \tag{4.11}$$

Note that the signs $-,-$ in the superscript (after the multiplicity label $\tilde{I}'$) on the l.h.s. of (4.6) indicate the signs of the extreme parameters $z''=z''_0$, $z=z_0$ as in Ališauskas (1978, 1982, 1988), in contrast to Ališauskas (1983) and the general case considered in this paper. The symmetry relation

$$\begin{bmatrix} (a'b') & (a''b'') & {}^{\tilde{I}',-,-}(a\ b) \\ (z')\,i' & (-i''_0)\,i''_0 & (z)\,i \end{bmatrix}^{(3)}_q = (-1)^{a+z-I} q^{-3(a+2z)/2} \times \left(\frac{[a+1]}{[2I+1]}\right)^{1/2} \begin{bmatrix} (b'a') & (a\ b) & {}^{\tilde{I}',-,-}(a''b'') \\ (-z')\,i' & (z)\,i & (-i''_0)\,i''_0 \end{bmatrix}^{(3)}_q \tag{4.12}$$

may be also useful.

The recoupling structure (4.1) and other related by symmetries constructions correspond to six versions of the boundary conditions. For $u_q(3)$, two versions are sufficient to cover completely all the cases of the coproduct decomposition (cf $SU(3)$ case, Ališauskas 1978, 1988). The mutual expansion coefficients of the isofactors satisfying the different boundary conditions form the triangular matrices, which may be inverted analytically. For example, let us consider equation (4.1) with $\kappa = \gamma^{(-)} = [\gamma_2, \dots, \gamma_{n-2}]$ and $r = p$. We have fixed the summation parameters $\overline{p} = \underset{\sim}{p}$ and $\widetilde{\gamma} = \gamma^{(-)}$ and one of isofactors on the r.h.s. equal to 1. For $u_q(3)$, $\kappa = \widetilde{\gamma} = 0$, $\tilde{\Lambda} = \mu$, multiplicity label $\overline{p}'$ is not necessary and $u_q(2)$ recoupling coefficient on the r.h.s. is unity. Hence, equation (3.25) of Ališauskas and Smirnov (1994) allows us to express a partial case of (4.1) in terms of the multiplicity-free isofactor (2.11). Accepting $\mu' = \lambda'$ in the second step, we obtain the expansion coefficients of the isofactors with superscript $\rho,+,+$ in terms of the boundary isofactors determined by condition (3.5), i.e. the $q$-version of (2.11) of Ališauskas (1988), with the appearance of $q$-numbers, $q$-factorials and $q$-phase

$$\begin{aligned} \tfrac{1}{2}\{&\tilde{i}(\tilde{i}+1) - \tilde{z}(\tilde{z}+1) - \tilde{I}'(\tilde{I}'+1) + \tilde{z}'(\tilde{z}'+1) + a'(a'+2) - a(a+2) \\ &+ b'(b'+1) - b(b+1) + a''(a-b'+2) + b''(b-a'+1) \\ &+ v(a''+b''-2a'-2a+b'+b-1)\}. \end{aligned} \tag{4.13}$$

For the inverse expansion, we use the same $q$-version of (2.11) of Ališauskas (1988), together with the symmetry relation

$$\begin{bmatrix} (a'b') & (a''b'') & {}^{\tilde{I}',-,-}(a\ b) \\ (z')\,i' & (z'')i'' & (z)\,i \end{bmatrix}^{(3)}_q = (-1)^{(a''-a)/2+\tilde{I}+i+z''-i'} q^{3(z''+a''/2)} \times \left(\frac{[a+1][2i'+1]}{[2\tilde{I}'+1][2i+1]}\right) \begin{bmatrix} (a\ b) & (b''a'') & {}^{-,+,\tilde{I}'}(a'b') \\ (z)\,i & (-z'')i'' & (z')\,i' \end{bmatrix}^{(3)}_q \tag{4.14}$$

(cf (2.21$b$) of Ališauskas 1988 and Klimyk 1993). In analogy with ratio of (2.19$c$) and (2.19$b$) of Ališauskas (1988), we may derive the boundary values of the $u_q(3)$ isofactors with superscript $-,+,\tilde{i}$, as well as the expansion coefficients of the linearly dependent $q$-isofactors with subscript $-,+,\tilde{i}$ (i.e. determined by condition (3.4) in previous section) in terms of the linearly independent $q$-isofactors labelled by subscript $-,+,\tilde{I}$, with an additional $q$-power

$$q^{[\tilde{i}(\tilde{i}+1)-\tilde{I}'(\tilde{I}'+1)]/2} \tag{4.15}$$

after the appearance of the corresponding $q$-numbers and $q$-factorials.

However, even all six versions of the boundary restrictions may be insufficient to cover all the cases of $u_q(4)$ isofactors, which will be discussed in the appendix.

## 5. Complementary $q$-algebras and $q$-recoupling coefficients, related to the overlaps of the coupled states

When the second irrep in the $u_q(n)$ coupling coefficient is a two-parametric (covariant and mixed tensor) representation and in the general $u_q(3)$ case, the overlaps of the coupled biorthogonal states may be expressed in terms of the explicit $q$-recoupling coefficients. In the case of the two-parametric covariant representation, they are equivalent to the resubducing coefficients (expressed in terms of the matrix elements of projectors) of the complementary (see Smirnov and Tolstoy 1992, Quesne 1992, Malashin *et al* 1994) chains of $q$-algebras. For this purpose, the $q$-boson realizations of the basis states of the $k$-parametric irreps of $u_q(n) \supset u_q(n-1) \supset u_q(n-2) \supset \cdots \supset u_q(2) \supset u_q(1)$ (which coincide with the basis of the consequently decomposed coproduct of $n$ symmetric irreps of $u_q(k)$) should be considered, as well as the basis states for chain $u_q(n) \supset u_q(n-2) \oplus u_q(2) \supset \cdots$ (which coincide with the coupled basis of $n$ symmetric irreps of $u_q(k)$ when two last irreps are coupled previously). The transformation brackets between different $u_q(n)$ bases (resubducing coefficients) coincide with corresponding recoupling coefficients of $u_q(k)$, as well as they coincide with the (re)subduction coefficients of the different bases of the braid groups and Hecke algebras (Pan and Chen 1993b, Pan 1993), taking into account the Schur–Weyl duality between $u_q(n)$ and $H_f(q)$.

The bilinear combinations of the $q$-recoupling coefficients (3.8$b$) coincide with the resubducing coefficients between the chains of the quantum algebras $u_q(k) \supset u_q(k-2) \oplus u_q(2)$ and $u_q(k) \supset u_q(k-1) \supset u_q(k-2)$, which may be expressed as the matrix elements

$$\sum_{\rho} U_n\big(\lambda'\; p_{k-1}\, {}^{\rho}\lambda\; p_k;\; \Lambda\,\varepsilon\big)_q\; U_n\big(\lambda'\; p'_{k-1}\varepsilon_2\, {}^{\rho}\lambda\; p'_k;\; \mu\,\varepsilon\big)_q$$

$$= \left\langle \begin{matrix} \lambda \\ \Lambda \\ \lambda' \end{matrix} \left| P^{[\varepsilon_1\varepsilon_2]q}_{p_{k-1},p_k;p'_{k-1},p'_k} \right| \begin{matrix} \lambda \\ \mu \\ \lambda' \end{matrix} \right\rangle_q \tag{5.1}$$

of the projection operator of the $u_q(2)$ subalgebra in the Löwdin–Shapiro form

$$P^{[\varepsilon_1\varepsilon_2]q}_{p_{k-1},p_k;p'_{k-1},p'_k} = \left(\frac{[p_k-\varepsilon_2]![p'_k-\varepsilon_2]!}{[\varepsilon_1-p_k]![\varepsilon_1-p'_k]!}\right)^{1/2} \sum_r \frac{(-1)^r[\varepsilon_1-\varepsilon_2+1]}{[r]![\varepsilon_1-\varepsilon_2+r+1]!} e_{k-1k}^{r+\varepsilon_1-p_k} e_{kk-1}^{r+\varepsilon_1-p'_k} \tag{5.2a}$$

$$= \left(\frac{[\varepsilon_1-p_k]![\varepsilon_1-p'_k]!}{[p_k-\varepsilon_2]![p'_k-\varepsilon_2]!}\right)^{1/2} \sum_r \frac{(-1)^r[\varepsilon_1-\varepsilon_2+1]}{[r]![\varepsilon_1-\varepsilon_2+r+1]!} e_{kk-1}^{r+p_k-\varepsilon_2} e_{k-1k}^{r+p'_k-\varepsilon_2}. \tag{5.2b}$$

Here

$$p_{k-1}+p_k = p'_{k-1}+p'_k = \varepsilon_1+\varepsilon_2$$

$$p_k = \sum_i \lambda_i - \sum_j \Lambda_j \qquad p'_k = \sum_i \lambda_i - \sum_j \mu_j$$

$$p_{k-1} = \sum_j \Lambda_j - \sum_i \lambda'_i \qquad p_{k-1} = \sum_j \mu_j - \sum_i \lambda'_i .$$

The first form of projector (5.2$a$) is more convenient for our purpose as (5.2$b$) which includes the usual maximal projector $P_{k-1k}$ (Smirnov *et al* 1991a,b; see also (2.11) of Ališauskas and Smirnov 1994). Restricting the all partitions $\lambda$, $\lambda'$, $\Lambda$ and $\mu$ in (5.1) to such which include no more as $n$ parts and using (2.6), we write the following expression for the bilinear combinations of the $q$-recoupling coefficients:

$$\sum_{\rho} U_n\big(\lambda'\; p_1\, {}^{\rho}\lambda\; p_2;\; \Lambda\,\varepsilon\big)_q\; U_n\big(\lambda'\; p'_1\, {}^{\rho}\lambda\; p'_2;\; \mu\,\varepsilon\big)_q$$

$$= \left( \frac{d_n[\Lambda] d_n[\mu] [p_2 - \varepsilon_2]! [p_2' - \varepsilon_2]!}{[\varepsilon_1 - p_2]! [\varepsilon_1 - p_2']!} \right)^{1/2} \frac{S_{n,n}[\Lambda; \lambda'] \, S_{n,n}[\mu; \lambda']}{S_{n,n}[\lambda; \Lambda] \, S_{n,n}[\lambda; \mu]}$$
$$\times \sum_{\sigma} (-1)^r \frac{[\varepsilon_1 - \varepsilon_2 + 1][r + \varepsilon_1 - p_2]! [r + \varepsilon_1 - p_2']! d_n[\sigma] S^2_{n,n}[\lambda; \sigma]}{[r]! \, [\varepsilon_1 - \varepsilon_2 + r + 1]! \, S^2_{n,n}[\Lambda; \sigma] \, S^2_{n,n}[\lambda; \sigma] \, S^2_{n,n}[\sigma; \lambda']} \qquad (5.3)$$

where

$$r = p_2 - \varepsilon_1 + \sum_{i=1}^{n} \Lambda_i - \sum_{j=1}^{n} \sigma_j$$

and $S_{n,n}[\ldots, \ldots]$ is defined as (2.7). The summation parameters $\sigma_j$ ($j = 1, 2, \ldots, n$) accept values in the region

$$\max(\lambda_j', \lambda_{j+1}) \leqslant \sigma_j \leqslant \min(\Lambda_j, \mu_j).$$

We see that the bilinear combinations (5.3) satisfy the Regge-type symmetry (up to elementary factors) with respect to the permutations of the arbitrary pairs of the sum restricting parameters.

Comparing the matrix elements of the projection operators in the both forms, we obtain the identity

$$U_n\big(\lambda' \; p_1 \; {}^{\rho}\lambda \; p_2; \; \Lambda \, \varepsilon\big)_q = U_n\big(\lambda^* \; p_1 \; {}^{\rho^*}\lambda'^* \; p_2; \; \Lambda^* \, \varepsilon\big)_q. \qquad (5.4)$$

Returning to the relation (3.2$b$) and substituting (3.8$a$) and (3.8$b$), we take in (5.3) $p_1 = \varepsilon_2 = p_1'$, $p_2 = p_2' = \varepsilon_1$, $\mu_n = \Lambda_n = \lambda_n' = 0$ and have $\sigma_n = 0$. Hence, the partial case of equation (5.2) may be simplified to $n - 1$ sums. In general, the expression (5.3) is $n$-independent when all its partitions contain less than $n$ parts.

It is expedient to use equation (4.3) with $\gamma = [a, 0]$, because we have the following expression for the bilinear combinations of the recoupling coefficients:

$$\sum_{\rho} U_n\big(\lambda', [\dot{0}, -p_n], {}^{\rho}\lambda, \; p_1; \Lambda, [a, \dot{0}, -b]\big)_q \; U_n\big(\lambda', [\dot{0}, -\tilde{p}_n], \; {}_{\rho}\lambda, \; p_1; \; \widetilde{\Lambda}, \; [a, \dot{0}, -b]\big)_q$$
$$= \left( \frac{d_n[\Lambda] \, d_n[\widetilde{\Lambda}]}{[p_n - b]! \; [\tilde{p}_n - b]! \; [p_1 + b + n - 1]! \; [\tilde{p}_1 + b + n - 1]!} \right)^{1/2}$$
$$\times [a + b + n - 1] \, S_{n,n}[\lambda'; \Lambda] \, S_{n,n}[\lambda'; \widetilde{\Lambda}] \, S_{n,n}[\lambda; \Lambda] \, S_{n,n}[\lambda'; \widetilde{\Lambda}]$$
$$\times \sum_{\sigma} (-1)^z \frac{[p_n - b + z]! \, [\tilde{p}_n - b + z]! \, [a + b + n - 2 - z]! \, d_n[\sigma]}{[z]! \; S^2_{n,n}[\sigma; \Lambda] \; S^2_{n,n}[\sigma; \widetilde{\Lambda}] \; S^2_{n,n}[\lambda'; \sigma] \; S^2_{n,n}[\lambda; \sigma]} \qquad (5.5)$$

where

$$p_1 - p_n = \tilde{p}_1 - \tilde{p}_n = a - b \qquad z = b + \sum_{i=1}^{n} (\sigma_i - \lambda_i)$$

and $\Lambda$, $\widetilde{\Lambda}$, $\lambda$, $\lambda'$ are sets of $n$ non-increasing integers, *differing* from the partitions denoted by the same letters in (4.1) and (4.3), but satisfying the betweenness conditions. The summation parameters $\sigma_i$ are changing in the intervals

$$\max(\widetilde{\Lambda}_i, \Lambda_i) \leqslant \sigma_i \leqslant \min(\lambda_i, \lambda_i').$$

Equation (5.5) is derived from an alternative version of (5.3) after the substitution (hook permutation, cf Ališauskas 1978, 1983)

$$\begin{aligned} &\lambda \to [\lambda_2, \lambda_3, \ldots, \lambda_n, \lambda_1 + n] \qquad &[\dot{0}, -p_n] \to [-p_n - n, \dot{0}] \\ &[a, \dot{0}, -b] \to [-b - n, a, \dot{0}] \qquad &[\dot{0}, -\tilde{p}_n] \to [-\tilde{p}_n - n, \dot{0}]. \end{aligned} \qquad (5.6)$$

In correspondence with our construction (4.1) of the isofactors satisfying elementary boundary conditions (4.2), we take in (5.5) $a = p_1 = \tilde{p}_1$, $b = p_n = \tilde{p}_n$, $\Lambda_n = \tilde{\Lambda}_n = \lambda_n$ and $\sigma_n = \lambda_n$ is also fixed. Thus, we again have expressions with $n-1$ summation parameters for overlaps of special $u_q(n)$ isofactors. Equations (5.3) and (5.5) cover the all cases of overlaps, necessary for the orthogonalization of the $u_q(3)$ isofactors.

We see that the corresponding $q$-recoupling coefficients (5.3) and (5.5) may be expressed without the $q$-phases after changing the usual factorials and representation dimension factors in the $U(n)$ recoupling coefficients (Bališauskas 1972, 1978, 1983) by $q$-factorials and the $q$-numbers, respectively, when definitions (2.3$a$) and (2.3$b$) are used.

Hence, we may solve the equations for the boundary $u_q(3)$ orthonormal isofactors of the type (3.8$a$) and orthogonalization coefficients in analogy with the explicit Gram–Schmidt procedure used for the paracanonical coupling coefficients of $SU(3)$ (Ališauskas 1988, 1990), without the appearance of the summation-parameter-dependent $q$-phases in their $q$-polynomial structure.


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### Appendix. Non-standard construction of isofactors for $u_q(4)$

Solutions of the boundary value problem related to (4.1) and (4.4) by the isofactor symmetries are possible for the $u_q(4)$ isofactors with different versions of multiplicity labelling of the resulting irrep $\lambda$ in the coproduct $\lambda' \otimes \lambda''$ decomposition (where $\lambda' = [\lambda'_1,\lambda'_2,\lambda'_3,0]$, $\lambda'' = [\lambda''_1,\lambda''_2,\lambda''_3,0]$) when the parameters of the irreps satisfy the following conditions:

$$\begin{array}{llllll}
\lambda''_3 \geqslant \lambda_4 & \text{or} & \lambda_3 \geqslant \lambda' & \text{for label} & \rho,+,+ \\
\lambda'_3 \geqslant \lambda_4 & \text{or} & \lambda_3 \geqslant \lambda'' & \text{for} & +,\rho,+ \\
\lambda_1 \geqslant \lambda'_1+\lambda''_2 & \text{or} & \lambda''_2 \geqslant \lambda_2 & \text{for} & \rho,-,- \\
\lambda_1 \geqslant \lambda'_2+\lambda''_1 & \text{or} & \lambda'_2 \geqslant \lambda_2 & \text{for} & -,\rho,- \\
\lambda'_3+\lambda''_1 \geqslant \lambda_1 & \text{or} & \lambda_4 \geqslant \lambda''_2 & \text{for} & +,-,\rho \\
\lambda'_1+\lambda''_3 \geqslant \lambda_1 & \text{or} & \lambda_4 \geqslant \lambda'_2 & \text{for} & -,+,\rho .
\end{array} \tag{A.1}$$

Therefore, for parameters in the region $\lambda''_2 \geqslant \lambda_4 \geqslant \lambda''_3$ we may use (and, of course, for $\lambda''_2 > \lambda_4 > \lambda''_3$ we *require*) the following construction of non-orthogonal $u_q(4)$ isofactors:

$$\begin{aligned}
\begin{bmatrix} \lambda' & \lambda'' & {}^{\{\tau',\Lambda,\tau''\}}\lambda \\ \mu' & \mu'' & {}_{+,-,\overline{\rho}}\mu \end{bmatrix}^{(4)}_q &= \sum_{\kappa',\kappa'',\nu,\overline{\rho}',\overline{\rho}'',\overline{\rho}_0} \begin{bmatrix} [\lambda_4,\lambda_4,\lambda''_3] & [\lambda''_1-\lambda_4,\lambda''_2-\lambda_4] & \lambda'' \\ [\lambda_4,\kappa'] & \kappa'' & {}_{+,-,\overline{\rho}_0}\mu'' \end{bmatrix}^{(4)}_q \\
&\quad \times \begin{bmatrix} \lambda' & [\lambda_4,\lambda_4,\lambda''_3] & {}^{+,-,\tau'}[\Lambda,\lambda_4] \\ \mu' & [\lambda_4,\kappa'] & {}_{+,-,\overline{\rho}'}\nu \end{bmatrix}^{(4)}_q \begin{bmatrix} [\Lambda,\lambda_4] & [\lambda''_1-\lambda_4,\lambda''_2-\lambda_4] & {}^{+,-,\tau''}\lambda \\ \nu & \kappa'' & {}_{+,-,\overline{\rho}''}\mu \end{bmatrix}^{(4)}_q \\
&\quad \times U_3\big(\mu', [\lambda_4,\kappa'], {}_{+,-,\overline{\rho}}\mu^{+,-,\overline{\rho}''}, \kappa;\ {}^{+,-,\overline{\rho}'}\nu,\ {}^{+,-,\overline{\rho}_0}\mu''\big)_q
\end{aligned} \tag{A.2}$$

where the diversity of the coupled states (exceeding, in general, the external multiplicity of $\lambda$ in the coproduct $\lambda' \otimes \lambda''$ decomposition) is generated by the partition $\Lambda = [\Lambda_1, \Lambda_2, \Lambda_3]$ of the intermediate irrep $[\Lambda,\lambda_4]$ (two degrees of freedom) and two auxiliary multiplicity labels $\tau' = [\tau'_1, \tau'_2]$ and $\tau'' = [\tau''_1, \tau''_2]$ (with a single degree of freedom each). All three isofactors on the r.h.s. are semistretched (the second one only after application of some symmetry property).

In order to find the linearly independent system between structures (A.2), we take $\mu'=\lambda'$, $\mu''=\lambda''^{(-)}=[\lambda''_2,\lambda''_3]$ and have fixed some summation parameters $\kappa'=[\lambda''_3,0]$, $\kappa''=[\lambda''_2-\lambda_4,0]$, $\nu=\Lambda$. The first isofactor on the r.h.s. is equal to 1 (multiplicity label $\overline{\rho}_0$ is absent) and $\delta_{\overline{\rho}',\tau'}$ appears in accordance with definition (3.5) instead of the second isofactor on the r.h.s. We reduce all the rows in partitions $\Lambda$, $\lambda$ and $\mu$ of the last isofactor by $\lambda_4$ and apply the inverted relation (3.2*b*); later we similarly reduce some partitions in the $u_q(3)$ recoupling coefficient, again apply (3.2*b*) and obtain

$$\begin{bmatrix} [\Lambda,\lambda_4] & [\lambda''_1-\lambda_4,\lambda''_2-\lambda_4] & {}^{+,-,\tau''}\lambda \\ \Lambda & [\lambda''_2-\lambda_4,0] & \mu \end{bmatrix}_q^{(4)} = q^{-\frac{1}{2}(\Lambda_3-\lambda_4)\sum_{i=1}^{3}(\lambda_i-\mu_i)}$$
$$\times\begin{bmatrix} [\Lambda_1-\Lambda_3,\Lambda_2-\Lambda_3] & [\lambda''_1-\lambda_4,\lambda''_2-\lambda_4] & {}^{+,-,\tilde{\tau}''}[\lambda_1-\Lambda_3,\lambda_2-\Lambda_3,\lambda_3-\Lambda_3] \\ [\Lambda_1-\Lambda_3,\Lambda_2-\Lambda_3] & [\lambda''_2-\lambda_4,0] & [\mu_1-\Lambda_3,\mu_2-\Lambda_3,\mu_3-\Lambda_3] \end{bmatrix}_q^{(4)}$$
$$\times\left(\frac{M[\mu_1-\lambda_4,\mu_2-\lambda_4,\mu_3-\lambda_4]\,M[\lambda_1-\Lambda_3,\lambda_2-\Lambda_3,\lambda_3-\Lambda_3]}{M[\mu_1-\Lambda_3,\mu_2-\Lambda_3,\mu_3-\Lambda_3]\,M[\lambda_1-\lambda_4,\lambda_2-\lambda_4,\lambda_3-\lambda_4]}\right)^{1/2} \tag{A.3}$$

where $\tilde{\tau}''=[\tau''_1-\Lambda_3,\tau''_2-\Lambda_3]$.

Using simpler notation and related to the construction of (4.1), we may write

$$\begin{bmatrix} [\Lambda'_1,\Lambda'_2] & [\varepsilon_1,\varepsilon_2] & {}^{+,-,\tau}\overline{\lambda} \\ [\Lambda'_1,\Lambda'_2] & [\varepsilon_2,0] & \mu \end{bmatrix}_q^{(4)} = \left(\begin{bmatrix} [\Lambda'_2,\Lambda'_2] & [\varepsilon_1,\tau_1-\Lambda'_2,\tau_2-\Lambda'_2] & \overline{\lambda} \\ [\Lambda'_2,\Lambda'_2] & [\tau_1-\Lambda'_2,\tau_2-\Lambda'_2] & \tau \end{bmatrix}_q^{(4)}\right.$$
$$\left.\times\begin{bmatrix} [\Lambda'_1,\Lambda'_2] & [\varepsilon_2] & \mu \\ [\Lambda'_1,\Lambda'_2] & 0 & [\Lambda'_1,\Lambda'_2] \end{bmatrix}_q^{(3)}\right)^{-1}\begin{bmatrix} [\Lambda'_2,\Lambda'_2] & [\varepsilon_1,\tau_1-\Lambda'_2,\tau_2-\Lambda'_2] & \overline{\lambda} \\ [\Lambda'_2,\Lambda'_2] & [\tau_1-\Lambda'_2,\tau_2-\Lambda'_2] & \mu \end{bmatrix}_q^{(4)}$$
$$\times\begin{bmatrix} [\Lambda'_1-\Lambda'_2] & [\varepsilon_2] & [\tau_1-\Lambda'_2,\tau_2-\Lambda'_2] \\ [\Lambda'_1-\Lambda'_2] & 0 & [\Lambda'_1-\Lambda'_2] \end{bmatrix}_q^{(3)}$$
$$\times\begin{bmatrix} [\Lambda'_2,\Lambda'_2] & [\tau_1-\Lambda'_2,\tau_2-\Lambda'_2] & \mu \\ [\Lambda'_2,\Lambda'_2] & [\Lambda'_1-\Lambda'_2] & [\Lambda'_1,\Lambda'_2] \end{bmatrix}_q^{(3)} \tag{A.4}$$

omitting isofactors equal to 1 of $u_q(4)$ and $u_q(3)$ for the coupling $[\Lambda'_2,\Lambda'_2]\otimes[\Lambda'_1-\Lambda'_2]$ to $[\Lambda'_1,\Lambda'_2]$ and $[\Lambda'_1-\Lambda'_2]\otimes[\varepsilon_1,\varepsilon_2]$ to $[\varepsilon_1,\tau_1-\Lambda'_2,\tau_2-\Lambda'_2]$. All the isofactors on the r.h.s. are known, with exception of the middle one which, after change of parameters to

$$\begin{bmatrix} [\Lambda_1,\Lambda_1] & [\lambda''_1,\lambda''_2,\lambda''_3] & \overline{\lambda} \\ [\Lambda_1,\Lambda_1] & [\lambda''_2,\lambda''_3] & \mu \end{bmatrix}_q^{(4)}$$

may be expressed by means of (3.2*b*) in terms of the $u_q(3)$ recoupling coefficients of the type

$$U_3\big([\Lambda_1,\Lambda_1],[\lambda''_2,\lambda''_3],\ \overline{\lambda},\ \lambda''_1;\ \mu,\ \lambda''\big)_q = (-1)^{\lambda''_2-\lambda''_3+\lambda_2-\mu_2}$$
$$\times\left(\frac{d_3[\mu]\,d_3[\lambda'']}{d_3[\overline{\lambda}]\,d_3[\lambda''_2,\lambda''_3]}\right)^{1/2} U_3\big(\lambda''_1,\ \mu,\ \lambda'',[0,-\Lambda_1,-\Lambda_1];\ \overline{\lambda},\ [\lambda''_2,\lambda''_3]\big)_q. \tag{A.5}$$

Here using notation $[0,-\Lambda_1,-\Lambda_1]$ for the symmetric covariant irrep $\Lambda_1$, we escape shifts in the remaining irrep denoting partitions. The phase factor in (A.5) corresponds to the $u_q(3)$ isofactor symmetries, correlated with the $SU(3)$ case (Pluhař *et al* 1986). The recoupling coefficient on the r.h.s. may be expressed without sum by means of (2.19*b*), taking into account the conjugation symmetry (2.19*a*).

The $u_q(3)$ recoupling coefficient on the r.h.s. of the boundary version of (A.2) may be written in analogy with (3.1) and (3.2) as a trivial sum:

$$U_3\big(\lambda', [\lambda_4,\lambda_3''], {}_{+,-,\sigma}\mu, [\lambda_2''-\lambda_4]; {}^{+,-,\tau'}\Lambda, [\lambda_2'',\lambda_3'']\big)_q$$
$$= \sum \left[\begin{array}{ccc} \lambda' & [\lambda_4,\lambda_3''] & {}^{+,-,\tau'}\Lambda \\ \overline{\lambda'} & \lambda_3'' & \sigma \end{array}\right]^{(3)}_q \left[\begin{array}{ccc} \Lambda & [\lambda_2''-\lambda_4] & \mu \\ \sigma & 0 & \sigma \end{array}\right]^{(3)}_q$$
$$\times \left[\begin{array}{ccc} [\lambda_4,\lambda_3''] & [\lambda_2''-\lambda_4] & [\lambda_2'',\lambda_3''] \\ \lambda_3'' & 0 & \lambda_3'' \end{array}\right]^{(3)}_q \qquad \text{(A.6)}$$

where $\sigma_1+\sigma_2=\lambda_1'+\lambda_2'+\lambda_3''$. Special $u_q(3)$ isofactor for coupling of $\Lambda \otimes [\lambda_2'',\lambda_3'']$ to $\mu$ on the l.h.s. is included, factually, in the subscript $+,-,\sigma$ of the non-orthonormal coupled state, according to (3.4) (but it should be not omitted in the multiplicity-free case). The first isofactor on the r.h.s. may be expressed by means of the generalized to $u_q(3)$ (see the end of section 4) expression, represented by ratio of (2.19*a*) and (2.19*b*) of Ališauskas (1988). Thus, the structure of (A.6) is more complicated when $\sigma$ accepts more values than the multiplicity of $\Lambda$ in coproduct $\lambda' \otimes [\lambda_4,\lambda_3'']$ decomposition.

Finally, we obtain the following expression for the boundary values of (A.2) (vanishing for $\mu_2 < \Lambda_2$ or $\mu_3 < \Lambda_3$):

$$\left[\begin{array}{ccc} \lambda' & \lambda'' & {}^{\{\tau',\Lambda,\tau''\}}\lambda \\ \overline{\lambda}' & \lambda''^{(-)} & {}_{+,-,\sigma}\mu \end{array}\right]^{(4)}_q = (-1)^{\mu_3-\Lambda_3} q^{\varrho_A}$$
$$\times \frac{[\tau_1''-\tau_2''+1]\, S_{3,3}[\overline{\lambda};\mu]\, S_{3,2}[\mu;\sigma]\, [\Lambda_1-\mu_2]!\, [\Lambda_1-\mu_3+1]!}{S_{3,2}[\overline{\lambda};\tau'']\, S_{3,2}[\Lambda;\sigma]\, S^2_{2,3}[\tau'';\mu]\, [\mu_1-\Lambda_1]!}$$
$$\times \left(\frac{d_3[\mu]\, [\lambda_2''-\lambda_4]!\, [\Lambda_2-\Lambda_3]!\, [\tau_1''-\Lambda_1]!}{[\Lambda_1-\Lambda_3+1]! [\Lambda_1-\tau_2'']! [\tau_2''-\Lambda_2]! [\tau_1''-\Lambda_2+1]!}\right.$$
$$\left.\times \prod_{i=1}^{3} \frac{[\mu_i-\lambda_4-i+3]!}{[\lambda_i-\lambda_4-i+3]!}\right)^{1/2} \left[\begin{array}{ccc} \lambda' & [\lambda_4,\lambda_3''] & {}^{+,-,\tau'}\Lambda \\ \overline{\lambda'} & \lambda_3'' & \sigma \end{array}\right]^{(3)}_q \qquad \text{(A.7)}$$

where

$$\tau_1'+\tau_2'=\sigma_1+\sigma_2=\lambda_1'+\lambda_2'+\lambda_3'' \qquad \tau_1''+\tau_2''=\Lambda_1+\Lambda_2+\lambda_2''-\lambda_4$$

$$\mu_1+\mu_2+\mu_3=\sum_{i=1}^{3}\Lambda_i+\lambda_2''-\lambda_4=\sum_{i=1}^{4}\lambda_i-\lambda_1''=\sum_{i=1}^{3}\lambda_i'+\lambda_2''+\lambda_3''$$

$$\begin{aligned}\varrho_A = \tfrac{1}{2}\{&(\mu_3-\Lambda_3)(\mu_2-\lambda_2-\Lambda_4+4)+(\mu_1-\Lambda_1)(\mu_2-\Lambda_1-\Lambda_2)\\ &-(\mu_2-\Lambda_2)(\Lambda_2-1)+(\tau_2''-\mu_2)(\tau_2''+\lambda_2''-\lambda_2-\lambda_4-2\Lambda_3+\Lambda_2+\mu_2+2)\\ &+(\tau_1''-\mu_1)(\lambda_2-\Lambda_1+2)+(\lambda_2''-\lambda_4)(\lambda_3'+\lambda_4)-(\Lambda_3-\lambda_4)(\lambda_1''+\lambda_4)\}\end{aligned}$$

and the boundary isofactor of $u_q(3) \supset u_q(2)$ on the r.h.s. of (A.7) is equal to $\delta_{\tau',\sigma}$, unless $\lambda_3'+\lambda_3''>\Lambda_3$ and $\Lambda_1>\lambda_2'+\lambda_4$, when its non-vanishing exceptional values (discussed at the end of section 4) may appear additionally for

$$\sigma_1 < \min(\Lambda_1-\lambda_4+\lambda_3'', \lambda_1'+\lambda_3'+\lambda_3''-\Lambda_3).$$

We obtain the complete set of the linearly independent non-orthonormal isofactors (A.2) for $u_q(4) \supset u_q(3)$ after restricting the parameters to

$$\tau_1' = \Lambda_1 \qquad \text{for } \tau_1''-\tau_1' \leqslant \lambda_2''-\lambda_4 \qquad \text{(A.8)}$$

and

$$\tau_2'' = \Lambda_2 \qquad \text{for } \tau_1'' - \tau_1' \geqslant \lambda_2'' - \lambda_4. \tag{A.9}$$

Really, we see that the indispensable conditions for non-vanishing of (A.7)

$$\tau_1'' \geqslant \mu_1 \qquad \tau_2'' \geqslant \mu_2 \qquad \tau_1' = \sigma_1 \qquad \text{(and sometimes } \tau_1' > \sigma_1)$$

give a distinctive distribution of zeros for each set $\tau'$, $\Lambda$, $\tau''$, satisfying condition (A.8) or (A.9). In addition to the betweenness conditions and other natural restrictions of parameters $\mu_i$, $\sigma_j$

$$\lambda_i \geqslant \mu_i \geqslant \lambda_{i+1} \qquad \mu_i \geqslant \sigma_i \geqslant \mu_{i+1} \qquad \mu_i \geqslant \lambda_i' \qquad \sigma_i \geqslant \lambda_i'$$

$$\lambda_1' + \lambda_2'' \geqslant \mu_1 \geqslant \lambda_3' + \lambda_2'' \geqslant \mu_3 \qquad \mu_1 \geqslant \lambda_2' + \lambda_3'' \geqslant \mu_3 \qquad \lambda_1' \geqslant \mu_3$$

$$\lambda_1' + \lambda_3'' \geqslant \mu_2 \geqslant \lambda_3' + \lambda_3'' \qquad \lambda_2' + \lambda_2'' \geqslant \mu_2$$

these external multiplicity distinguishing parameters should satisfy specified conditions (3.7*a*):

$$\begin{aligned} &\mu_3 \leqslant \lambda_4 + \lambda_3' \qquad \mu_3 - \mu_1 + \sigma_1 \leqslant \lambda_3 + \lambda_4 - \lambda_2'' \qquad \mu_1 \geqslant \lambda_1 - \lambda_1'' + \lambda_2'' \\ &\sigma_2 - \mu_3 \leqslant \lambda_2' - \lambda_3' \qquad \mu_1 - \sigma_1 \leqslant \lambda_2'' - \lambda_3''. \end{aligned} \tag{A.10}$$

Using the correspondence

$$\tau_1'' \leftrightarrow \mu_1 \qquad \tau_2'' \leftrightarrow \mu_2 \qquad \tau_1' \leftrightarrow \sigma_1 \tag{A.11}$$

together with (A.8) or (A.9), we verified that the inverted Littlewood–Richardson conditions (A.10) are in one-to-one correspondence with the conditions for $\tau'$ and $\tau''$ (cf (3.7*b*), or (2.3) of Ališauskas 1988). Hence, we demonstrated the completeness of (A.2) under restrictions (A.8) or (A.9). In order to obtain the explicit isofactors satisfying the boundary condition (3.5), we may try to invert analytically the corresponding triangular expansion matrices (A.7), beginning from the diagonal with respect to $\tau_1'$ and $\sigma_1$ submatrices.

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